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J. Math. Anal. Appl. 284 (2003) 698–710

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Unique continuation for solutions to Maxwell's system with non-analytic anisotropic coefficients

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Received 11 September 2002

Submitted by C.E. Wayne

Abstract

We consider Maxwell's system with anisotropic coefficients, i.e., the electric permittivity and the magnetic permeability are assumed to be matrices with non-analytic entries. Under the additional assumption that one matrix is a scalar multiple of the other we prove unique continuation of the system across non-characteristic surfaces. The proof relies on differential geometry as well as on Carleman estimates.

1. Introduction and main result

Let $\Omega \subset \mathbf{R}^3$ be an open set with a C^∞ boundary $\partial\Omega$. The evolution of the electromagnetic field in Ω over a finite time interval $(0, T)$ is modeled by Maxwell's equations

$$\begin{aligned} &\varepsilon(x)\partial_t E(t,x) - \operatorname{curl} H(t,x) = 0, \\ &\mu(x)\partial_t H(t,x) + \operatorname{curl} E(t,x) = 0 \quad \text{in } Q = (0,T)\times\Omega, \\ &\operatorname{div}\big(\varepsilon(x)E(t,x)\big) = \operatorname{div}\big(\mu(x)H(t,x)\big) = 0, \end{aligned} \tag{1.1}$$

where E and H denote the electric field intensity and the magnetic field intensity, respectively. Moreover, ε and μ are coefficient functions describing the electric permittivity and the magnetic permeability. They are assumed to be 3×3 positive definite, symmet-

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[1] Research of M.M. Eller supported by the National Science Foundation through Grant DMS-0049020.

0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00391-3

ric matrices whose entries are time-independent C^2 functions, i.e., $\varepsilon^{jk}, \mu^{jk} \in C^2(\bar{\Omega})$ for $1 \leqslant j, k \leqslant 3$.

In this paper we will study the unique continuation property of Maxwell's system. We will discuss uniqueness of the lateral Cauchy problem, i.e., under what conditions $E = H = 0$ on $(0, T) \times \partial\Omega$ implies $E = H = 0$ in Q for some $T > 0$. This problem is of great interest to applications in control theory and inverse problems. It has been previously considered in the case that ε and μ are positive, two times continuously differentiable scalar functions [1, Theorem 49], [2, Theorem 4.2]. In that special case one can diagonalize the system and obtain a system of six second order equations with the wave operator as the principal part in each component. Then one applies Carleman estimates for the wave operator and obtains a local uniqueness result.

The case of anisotropic coefficients has not been considered so far. We propose a new approach to the anisotropic system based on differential geometry. Assuming that the matrices ε and μ are multiples of each other, i.e., $\mu = f\varepsilon$ for some positive scalar function $f \in C^2(\bar{\Omega})$ we will be able to show that E and H satisfy a system of six second order hyperbolic equations. Again, as in the scalar case the system is only coupled through lower order terms. This allows us to apply Carleman estimates and to derive a local uniqueness result, Theorem 4.1. The local uniqueness result can be described as a local Holmgren theorem. This is to say that we obtain unique continuation across non-characteristic surfaces.

From the local uniqueness result we deduce the main result.

Theorem 1.1. *Suppose that ε is a positive definite, symmetric matrix such that $\varepsilon^{jk} \in C^2(\bar{\Omega})$, that $f \in C^2(\bar{\Omega})$ is a scalar, positive function, and that $\mu = f\varepsilon$. Let $\Gamma \subset \partial\Omega$ be non-empty and open. Furthermore, assume that $E \in \mathcal{D}'(0, T; L_2(\Omega)^3)$ and $H \in \mathcal{D}'(0, T; L_2(\Omega)^3)$ are solutions to Maxwell's system* (1.1) *and that $E = 0$ and $H = 0$ on $(0, T) \times \Gamma$. Then $E \equiv 0$ and $H \equiv 0$ on $\{T/2\} \times \Omega$ provided*

$$T > T_0 = 2 \sup_{x \in \Omega} \bigl\{ \operatorname{dist}_G(x, \Gamma) \bigr\}. \tag{1.2}$$

Here the distance $\operatorname{dist}_G(x, \Gamma)$ is induced by the Riemannian metric

$$G_{jk}(x) = f(x) \det\bigl(\varepsilon(x)\bigr) \varepsilon_{jk}(x),$$

where ε_{jk} denotes the entries of ε^{-1}.

Indeed, the proof of this theorem follows from our local uniqueness result, Theorem 4.1 in Section 4 by means of Theorem 1 in [5]. Even though Littman's paper deals with the wave equation, he writes [5, p. 367]:

> "... The method of proof of Theorems 1 and 2 can be extended to other situations as long as one has the local version of Holmgren's theorem."

This local version of Holmgren's theorem is furnished by Theorem 4.1 in our case. The achievement of Littman's paper is that he clarifies the relation between local and global uniqueness results. He shows that the argument which deduces global uniqueness based on local uniqueness is of purely geometric nature. The differential equation or in our case, the system of differential equations, does not appear in that argument. Hence, it works for both scalar equations and systems of equations.

If we complement Maxwell's system by a homogeneous boundary condition, then we obtain the following

Corollary 1.2. *Assume that all the assumptions of Theorem* 1.1 *are satisfied. Moreover, assume that either*

$$\nu \times E = 0 \quad \textit{or} \quad \nu \times H = 0 \quad \textit{on } (0,T) \times \partial\Omega, \tag{1.3}$$

and that $T > T_0$. Then $E \equiv 0$ and $H \equiv 0$ in $Q = (0,T) \times \Omega$.

Proof. By Theorem 1.1 we get $E \equiv 0$ and $H \equiv 0$ on $\{T/2\} \times \Omega$. Hence, E and H are solutions to Maxwell's system (1.1) with one of the boundary conditions in (1.3) and with zero initial conditions at $t = T/2$. This initial boundary value problem is uniquely solvable in time and we get $E \equiv 0$ and $H \equiv 0$ in $(T/2, T) \times \Omega$. The same can be done backward in time since Maxwell's system does not change in nature when we replace t by $-t$. □

This paper is structured as follows. In Section 2 we introduce some notions of differential geometry which we use in Section 3 for the diagonalization of Maxwell's system. Section 4 contains the local uniqueness theorem and its proof.

Throughout this paper we will use the following function spaces. The linear space of all complex-valued functions with j continuous derivatives in a set K is denoted by $C^j(K)$. We set $C^\infty(K) = \bigcap_{j=1}^\infty C^j(K)$ and by $C_0^\infty(K)$ we denote all the functions in $C^\infty(K)$ with support in some $K' \Subset K$. The dual space of $C_0^\infty(K)$ is $\mathcal{D}'(K)$, the space of distributions. The action of a distribution $u \in \mathcal{D}'(K)$ on a test function $\eta \in C_0^\infty(K)$ is denoted by $\langle u, \eta\rangle$. Similarly, the space of compactly supported distribution $\mathcal{E}'(K)$ is the dual space of $C^\infty(K)$. The linear space of all square integrable, Lebesgue measurable complex-valued functions on K is denoted by $L_2(K)$. By $H^s(K)$ we denote the L_2 based Sobolev of order s. The space $H^s_{\mathrm{loc}}(K)$ is the corresponding local Sobolev space, i.e., we say $u \in H^s_{\mathrm{loc}}(K)$ if $u \in H^s(K')$ for every $K' \Subset K$. If we consider a vector fields u with m components in a certain space X we will write $u \in X^m$. Some special function spaces will be introduced in Section 4.

2. Differential geometry

A Riemannian metric on an open subset M of $\mathbf{R}^n$ is given by $g_{jk}(x)$, where $g_{jk}(x)$ is a positive definite, symmetric $n \times n$ matrix function with C^∞ entries. The linear space of vector fields with C^∞ components is denoted by $\mathcal{X}(M)$. The dual space, i.e., the space of covector fields (1-forms) with C^∞ components is denoted by $\Lambda(M)$. For $C = C_j(x)\,dx^j \in \Lambda(M)$ and $A = A^j(x)\partial_j \in \mathcal{X}(M)$ we have $\langle C, A\rangle(x) = C_j(x)A^j(x)$. If $A, B \in \mathcal{X}(M)$ then their inner product is the scalar function

$$(A, B)(x) = g_{jk}(x)A^j(x)B^k(x).$$

Here and in the following we will use the Einstein summation convention. The metric $g_{jk}(x)$ assigns to every vector field A a covector field $A_\flat$ through the relation $\langle A_\flat, B\rangle = (A, B)$ for all vector fields B. Thus

$$A_\flat = A_k(x)\,dx^k, \quad \text{where } A_k(x) = g_{jk}(x)A^j(x).$$

This operation is also known as the index-lowering operation. Conversely, if we multiply a 1-form with the inverse matrix $g^{jk}(x)$ we obtain the corresponding vector field (index-raising operation). Given $C = C_j(x)\,dx^j$ we have

$$C^\sharp = C^k(x)\frac{\partial}{\partial x_k}, \quad \text{where } C^k(x) = g^{jk}(x)C_j(x). \tag{2.1}$$

The inner product of two 1-forms $A_\flat$ and $B_\flat$ is defined as

$$(A_\flat, B_\flat)(x) = g^{jk}(x)A_j(x)B_k(x).$$

We observe $(A, B) = (A_\flat, B_\flat)$.

The inner product of two k-forms

$$u = u_1 \wedge u_2 \wedge \cdots \wedge u_k \quad \text{and} \quad v = v_1 \wedge v_2 \wedge \cdots \wedge v_k,$$

where $u_1, u_2, \ldots, u_k$ and $v_1, v_2, \ldots, v_k$ are 1-forms is given by

$$(u, v) = \sum_\pi \operatorname{sgn}(\pi)(u_1, v_{\pi(1)})(u_2, v_{\pi(2)})\ldots(u_k, v_{\pi(k)}),$$

where π ranges over the set of permutations of $\{1, 2, \ldots, l\}$.

The Hodge star operator $*$ maps k forms into $n-k$ forms and is defined by the relation

$$u \wedge *v = (u, v)\omega \quad \text{for all } k\text{-forms } u, \tag{2.2}$$

where $\omega = g^{1/2}\,dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is the volume form on M. Here $g(x) = \det(g_{jk}(x))$.

The exterior derivative of the 1-form C is the 2-form

$$dC = \frac{\partial C_j}{\partial x_k}\,dx^k \wedge dx^j \tag{2.3}$$

and similarly the exterior derivative maps k forms into $k+1$ forms. The adjoint of the exterior derivative operator δ is defined by

$$(du, v) = (u, \delta v)$$

for all compactly supported k-forms u and $(k+1)$-forms v. Thus δ maps $(k+1)$-forms into k-forms. Then, the Hodge Laplacian is defined as

$$-\Delta = d\delta + \delta d.$$

In the case $n = 3$, the following formulas can be verified by computing both sides in (2.2):

$$\begin{aligned}
*dx^1 &= g^{\frac{1}{2}}\big[g^{11}\,dx^2 \wedge dx^3 + g^{12}\,dx^3 \wedge dx^1 + g^{13}\,dx^1 \wedge dx^2\big],\\
*dx^2 &= g^{\frac{1}{2}}\big[g^{21}\,dx^2 \wedge dx^3 + g^{22}\,dx^3 \wedge dx^1 + g^{23}\,dx^1 \wedge dx^2\big],\\
*dx^3 &= g^{\frac{1}{2}}\big[g^{31}\,dx^2 \wedge dx^3 + g^{32}\,dx^3 \wedge dx^1 + g^{33}\,dx^1 \wedge dx^2\big].
\end{aligned} \tag{2.4}$$

Furthermore,

$$\begin{aligned}
*(dx^1 \wedge dx^2) &= g^{\frac{1}{2}}\big[(g^{11}g^{22} - g^{12}g^{21})\,dx^3 + (g^{21}g^{32} - g^{22}g^{31})\,dx^1 \\
&\quad + (g^{12}g^{31} - g^{11}g^{32})\,dx^2\big], \\
*(dx^2 \wedge dx^3) &= g^{\frac{1}{2}}\big[(g^{12}g^{23} - g^{13}g^{22})\,dx^3 + (g^{22}g^{33} - g^{23}g^{23})\,dx^1 \\
&\quad + (g^{13}g^{23} - g^{12}g^{33})\,dx^2\big], \\
*(dx^3 \wedge dx^1) &= g^{\frac{1}{2}}\big[(g^{13}g^{21} - g^{11}g^{23})\,dx^3 + (g^{23}g^{31} - g^{21}g^{33})\,dx^1 \\
&\quad + (g^{23}g^{11} - g^{31}g^{13})\,dx^2\big],
\end{aligned} \tag{2.5}$$

and one obtains

$$\begin{aligned}
\big(*(dx^1 \wedge dx^2)\big)^\sharp &= g^{-\frac{1}{2}}\frac{\partial}{\partial x_3}, \\
\big(*(dx^2 \wedge dx^3)\big)^\sharp &= g^{-\frac{1}{2}}\frac{\partial}{\partial x_1}, \\
\big(*(dx^3 \wedge dx^1)\big)^\sharp &= g^{-\frac{1}{2}}\frac{\partial}{\partial x_2}.
\end{aligned} \tag{2.6}$$

Also,

$$*(dx^1 \wedge dx^2 \wedge dx^3) = g^{-\frac{1}{2}} \tag{2.7}$$

and $\delta = -{*d*}$ on 1-forms and $\delta = {*d*}$ on 2-forms.

Finally, we notice that most of the facts stated in this section are valid also for Riemannian metrics and vector fields with less regularity. In particular, we can assume that $g_{jk} \in C^1$ and that the vector and the covector fields are of regularity H^1. The index raising and the index lowering operations will be still well defined since the product of a C^1 function and an H^1 function is in H^1. Moreover, the exterior derivative of a 1-form with components in H^1 will be a 2-form with components in L_2. Furthermore the formulas involving the Hodge star operator, i.e., formulas (2.4)–(2.7) remain valid for $g^{jk} \in C^1$ and $g \in C^1$.

3. The diagonalization of Maxwell's system

We will diagonalize Maxwell's system under the assumption the $\mu = f\varepsilon$ for some positive scalar function $f \in C^2(\bar{\Omega})$. In order to make use of differential geometry outlined in the previous section we will need a certain regularity of the solutions to Maxwell's system. The following proposition clarifies that we can assume that our solutions are H^1 at least in the space variable.

Proposition 3.1. *Assume that $\varepsilon, \mu \in C^2(\bar{\Omega})$ and that $E \in \mathcal{D}'(0,T; L_2(\Omega)^3)$ and $H \in \mathcal{D}'(0,T; L_2(\Omega)^3)$ are solutions to Maxwell's system* (1.1). *Then $E \in \mathcal{D}'(0,T; H^1_{\mathrm{loc}}(\Omega)^3)$ and $H \in \mathcal{D}'(0,T; H^1_{\mathrm{loc}}(\Omega)^3)$.*

Proof. Solutions $E \in \mathcal{D}'(0,T; L_2(\Omega)^3)$ and $H \in \mathcal{D}'(0,T; L_2(\Omega)^3)$ are understood as

$$
\begin{aligned}
&-\varepsilon\langle E, \partial_t \eta\rangle - \langle \operatorname{curl} H, \eta\rangle = 0,\\
&-\mu\langle H, \partial_t \rho\rangle + \langle \operatorname{curl} E, \rho\rangle = 0 \quad \text{in } L_2(\Omega),\\
&\big\langle \operatorname{div}(\varepsilon E), \kappa\big\rangle = \big\langle \operatorname{div}(\mu H), \lambda\big\rangle = 0,
\end{aligned}
\tag{3.1}
$$

for all $\eta, \rho \in C_0^\infty(0,T)^3$ and all $\kappa, \lambda \in C_0^\infty(0,T)$. Since $\varepsilon\langle E, \partial_t\eta\rangle \in L_2(\Omega)^3$ we conclude $\langle \operatorname{curl} H, \eta\rangle \in L_2(\Omega)^3$ by the first equation in (3.1). Moreover, $\langle \operatorname{div} H, \lambda\rangle \in L_2(\Omega)$ by the third equation in (3.1). Since curl and div form an elliptic system of order one we conclude $\langle H, \eta\rangle \in H^1_{\mathrm{loc}}(\Omega)^3$ for all $\eta \in C_0^\infty(0,T)^3$. Of course, the same regularity result holds for E. $\square$

We interpret Ω as a Riemannian manifold with metric $g_{jk} = \varepsilon^{-1}$. In the following we interpret E and H as covector fields with H^1_{loc} components. Technically, we have to work with $\langle E, \eta\rangle \in H^1_{\mathrm{loc}}(\Omega)^3$ and $\langle H, \rho\rangle \in H^1_{\mathrm{loc}}(\Omega)^3$ but we will avoid this notation and write E and H instead.

Maxwell's system (1.1) appears as

$$
\begin{aligned}
&\partial_t E^\sharp - \operatorname{curl} H = 0,\\
&f\partial_t H^\sharp + \operatorname{curl} E = 0 \quad \text{in } Q = (0,T)\times\Omega,\\
&\operatorname{div} E^\sharp = \operatorname{div}(f H^\sharp) = 0,
\end{aligned}
\tag{3.2}
$$

when we use the index-lowering operation defined by formula (2.1). Note that

$$
\operatorname{curl} H = \left(\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1}\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}\right)\frac{\partial}{\partial x_3}.
$$

Using the definition of the exterior derivative (2.3) and formula (2.6) we get

$$
\operatorname{curl} H = g^{\frac{1}{2}}(*dH)^\sharp, \tag{3.3}
$$

where $g = \det(g_{jk}) = \det(\varepsilon^{-1})$. Using formula (3.3) we can represent the first two equations in (3.2) as

$$
\partial_t E^\sharp - g^{\frac{1}{2}}(*dH)^\sharp = 0, \qquad f\partial_t H^\sharp + g^{\frac{1}{2}}(*dE)^\sharp = 0.
$$

Differentiating the first equation with respect to t and applying the index-lowering operator gives

$$
\partial_t^2 E - g^{\frac{1}{2}} * d\partial_t H = 0. \tag{3.4}
$$

To the second equation we apply the index-lowering operator and the exterior derivative and get

$$
f\, d\partial_t H + df \wedge \partial_t H + dg^{\frac{1}{2}} \wedge *dE + g^{\frac{1}{2}} d * dE = 0.
$$

Next we apply the Hodge star operator and multiply by $g^{1/2}$ and obtain

$$
fg^{\frac{1}{2}} * d\partial_t H + g^{\frac{1}{2}} * (df \wedge \partial_t H) + g^{\frac{1}{2}} * (dg^{\frac{1}{2}} \wedge *dE) + g\delta dE = 0,
$$

where we also observed that $*d* = \delta$. Then we replace $\partial_t H$ by $-g^{1/2}/f * dE$ and get

$$
fg^{\frac{1}{2}} * d\partial_t H - g^{\frac{1}{2}} * \left(df \wedge \frac{g^{\frac{1}{2}}}{f} * dE\right) + g^{\frac{1}{2}} * (dg^{\frac{1}{2}} \wedge *dE) + g\delta\, dE = 0.
$$

Adding the last equation and equation (3.4) multiplied by f we have

$$f\partial_t^2 E + g\delta dE = -g^{\frac{1}{2}} * (dg^{\frac{1}{2}} \wedge *dE) + g^{\frac{1}{2}} * \left(df \wedge \frac{g^{\frac{1}{2}}}{f} * dE\right). \tag{3.5}$$

We notice that the term in the right-hand side consists only of one derivative in the components of E, in other words it is a lower order term.

Now we turn our attention to the divergence equation $\operatorname{div} E^\sharp = 0$. Clearly,

$$\operatorname{div} E^\sharp = \frac{\partial E^1}{\partial x_1} + \frac{\partial E^2}{\partial x_2} + \frac{\partial E^3}{\partial x_3}$$

but our goal is to write the divergence in terms of the operators of differential geometry introduced in Section 2. With the help of (2.4) we compute

$$*E = E_j * dx^j = g^{\frac{1}{2}}[E^1 dx^2 \wedge dx^3 + E^2 dx^3 \wedge dx^1 + E^3 dx^1 \wedge dx^2]$$

and applying the exterior derivative yields

$$d * E = \left(\frac{\partial(g^{\frac{1}{2}}E^1)}{\partial x_1} + \frac{\partial(g^{\frac{1}{2}}E^2)}{\partial x_2} + \frac{\partial(g^{\frac{1}{2}}E^3)}{\partial x_3}\right) dx^1 \wedge dx^2 \wedge dx^3.$$

Using formula (2.7) we obtain

$$*d*E = g^{-\frac{1}{2}}\left(\frac{\partial(g^{\frac{1}{2}}E^1)}{\partial x_1} + \frac{\partial(g^{\frac{1}{2}}E^2)}{\partial x_2} + \frac{\partial(g^{\frac{1}{2}}E^3)}{\partial x_3}\right) = \operatorname{div} E^\sharp + g^{-\frac{1}{2}}\frac{\partial g^{\frac{1}{2}}}{\partial x_j}E^j.$$

Thus, the equation $\operatorname{div} E^\sharp = 0$ implies

$$\delta E = -g^{-\frac{1}{2}}\frac{\partial g^{\frac{1}{2}}}{\partial x_j}E^j.$$

Together with (3.5) we obtain

$$\begin{aligned} f\partial_t^2 E + g(\delta d + d\delta)E &= -g^{\frac{1}{2}} * (dg^{\frac{1}{2}} \wedge *dE) + \frac{g}{f} * (df \wedge *dE) \\ &\quad - gd\left(g^{-\frac{1}{2}}\frac{\partial g^{\frac{1}{2}}}{\partial x_j}E^j\right). \end{aligned} \tag{3.6}$$

The last term in this equation gives the reason that we assume $g_{jk} \in C^2(\bar{\Omega})$. Now we use the fact that the Hodge Laplacian $\Delta = -(\delta d + d\delta)$ on k-forms is given by

$$\Delta = g^{jk}(x)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} + l^k(x)\frac{\partial}{\partial x_k} + l(x), \quad l^k, l \in L_\infty(\bar{\Omega}), \tag{3.7}$$

[8, Chapter 2, Section 10] and obtain a system of three second order differential equations for the components E_j of the one 1-form E. A crucial observation is that this system is coupled only through first order terms and zero order terms, more specifically we have

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}E_j(t,x) - \frac{g(x)}{f(x)}g^{km}(x)\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_m}E_j(t,x) \\ &\quad = p_j^{km}(x)\frac{\partial}{\partial x_k}E_m(t,x) + q_j^k(x)E_k(t,x) \end{aligned} \tag{3.8}$$

for $j = 1, 2, 3$, where p_j^{km} and q^{jk} are essentially bounded functions that account for the lower order terms in (3.6) and (3.7). With the same technique one derives an equation for the components of the 1-form H,

$$\frac{\partial^2}{\partial t^2} H_j(t,x) - \frac{g(x)}{f(x)} g^{km}(x) \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_m} H_j(t,x) = \tilde{p}_j^{km}(x) \frac{\partial}{\partial x_k} H_m(t,x) + \tilde{q}_j^k(x) H_k(t,x) \tag{3.9}$$

for $j = 1, 2, 3$. These equations are well known for the isotropic system, i.e., the case when ε and μ are positive scalar functions. We believe that these equations are new in the special anisotropic case which we consider here. However, there seems to be a connection to the electrodynamics in curved spacetime [6, Chapter 22].

4. Local uniqueness for Maxwell's system

In this section we prove a local uniqueness result for system (3.8)–(3.9). Of course, it will be sufficient only to show uniqueness for system (3.8) since (3.9) is of the same structure. We note that very similar theorems have been proved before [2, Theorem 3.2] and [1, Theorem 49]. However, both pertain to slightly different situation with respect to either the regularity of the solution or the structure of the differential operator. That is why we give a complete proof of the theorem below.

In the following we set

$$P(x, D) = \frac{\partial^2}{\partial t^2} - \frac{g(x)}{f(x)} g^{km}(x) \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_m}$$

and we assume that S is the level surface of a continuous function ϕ in Q, i.e., $S = \{\phi = 0\}$. The main result of this section is

Theorem 4.1. *Let S be a C^2-surface that is non-characteristic with respect to the operator $P(x, D)$ at $(t_0, x_0) \in Q$, i.e., $P(x_0, \nabla\phi(t_0, x_0)) \neq 0$. Assume that $E \in \mathcal{D}'(0, T; H^1_{\mathrm{loc}}(\Omega)^3)$ is a solution to* (3.8) *that vanishes on one side of S. Then $E \equiv 0$ in a full neighborhood of (t_0, x_0).*

Proof. The proof of this theorem is based on Carleman estimates in connection with a localization/perturbation argument. In order to formulate the Carleman estimate we introduce some notation. Corresponding to the time variable t and the space variable x we introduce Fourier variables ξ_0 and ξ' and set $\xi = (\xi_0, \xi')$. The Fourier transform of a function u is given by

$$\mathcal{F}[u](\xi) = \hat{u}(\xi) = \int_{\mathbf{R}^4} e^{-i(\xi_0 t + \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)} u(x)\, dx.$$

The Gaussian regularizer is the operator

$$e^{-\frac{1}{2\tau} D_t^2} u = \frac{1}{(2\pi)^4} \int_{\mathbf{R}^4} e^{-\frac{1}{2\tau}\xi_0^2} e^{i(\xi_0 t + \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)} \hat{u}(\xi)\, d\xi, \quad \tau > 0,$$

where we think about τ as a large, positive parameter. Also, we need some weighted norms on the Sobolev space H^s. We set

$$|u|_{s,\tau}^2 = \frac{1}{(2\pi)^4} \int \left(\tau^2 + |\xi|^2\right)^s \left|\hat{u}(\xi)\right|^2 d\xi$$

for $s \in \mathbf{R}$ and $\tau > 0$. Furthermore, for $N, s \in \mathbf{R}$, we introduce the space $H^{N,s}$ as space of all distributions for which the norm

$$|u|_{N,s,\tau} = \frac{1}{(2\pi)^2} \left(\int \left(\tau^2 + \xi_0^2\right)^N \left(\tau^2 + |\xi|^2\right)^s \left|\hat{u}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}}$$

is finite.

Step 1: The Carleman estimate. The following Carleman estimate for the operator $P(x, D)$ is due to Tataru [7, Theorem 1]. Also, for the notation of strong pseudo-convexity we refer to [7, Definition 1]. We shift the coordinate system such that $(t_0, x_0) = 0$.

Let ψ be a second degree polynomial that is strongly pseudo-convex with respect to $P(x, D)$ on the level surface $\{\xi_0 = 0\}$ at 0 and let N be a positive integer. Then there exists a constant $c > 0$ depending only on ψ and the operator P such that for small $\delta > 0$ and large τ the estimate

$$\begin{aligned} &\frac{1}{\tau}\left|e^{-\frac{1}{2\tau}D_t^2} e^{\tau\psi} u\right|_{2,\tau}^2 \\ &\quad \leqslant c\left(\left|e^{-\frac{1}{2\tau}D_t^2} e^{\tau\psi} P(x, D)u\right|_0^2 + e^{-\delta\tau}\left(|u|_{-N,1,\tau}^2 + \left|P(x, D)u\right|_{-N,0,\tau}^2\right)\right) \end{aligned} \tag{4.1}$$

holds provided u is a compactly supported distribution in $B_\delta(0)$ such that the right-hand side is finite. Here $B_\delta(0)$ denotes the open ball with radius δ and center at the origin.

Step 2: Perturbation of the initial surface. Since S is non-characteristic with respect to $P(x, D)$ at 0, the surface S is also strongly pseudo-convex with respect to $P(x, D)$ at 0 on the level surface $\{\xi_0 = 0\}$ [7, Section 5.1]. Furthermore, for sufficiently large $\lambda > 0$ the function

$$\varphi(t, x) = e^{\lambda\phi(t,x)} - 1$$

is strongly pseudo-convex with respect to the operator $P(x, D)$ at 0 on the level surface $\{\xi_0 = 0\}$ [4, Section 28.3]. Of course, the surface S is likewise given as a level surface of the function φ, i.e., $S = \{\varphi = 0\}$. We denote the two sides of S by S^+ and S^-, respectively, and we assume that $E = 0$ in $S^+ = \{\varphi > 0\}$ for $j = 1, 2, 3$.

Consider the following second degree polynomial in (t, x):

$$\psi(t, x) = \sum_{|\alpha| \leqslant 2} \frac{\partial^\alpha \varphi(0)}{\alpha!} (t, x)^\alpha - 3\epsilon\left(t^2 + |x|^2\right). \tag{4.2}$$

The geometry of the level surfaces of ψ can be described as follows. The level surface $\{\psi(t, x) = 0\}$ is contained in S^+ for $(t, x) \neq 0$ and $\delta > 0$ small enough. Moreover, the level surface $\psi(t, x) = -\beta$, $\beta > 0$ is contained in S^- for $t^2 + |x|^2$ small. A picture explaining the relation of the functions φ and ψ can be found in [3, p. 114].

Then we choose $\epsilon > 0$ such that ψ is still strongly pseudo-convex with respect to $P(x, D)$ at 0 on $\{\xi_0 = 0\}$ and that

$$\psi(t, x) \leqslant \varphi(t, x) - 2\epsilon\left(t^2 + |x|^2\right) \quad \text{for all } (t, x) \in B_{2\delta}(0).$$

This can be done since strong pseudo-convexity is stable under small perturbations and by choosing $\delta > 0$ small.

Step 3: Localization of E. Next we introduce a smooth cutoff function χ such that

$$\chi(s) = \begin{cases} 1 & \text{if } s \geqslant -\epsilon\delta^2, \\ 0 & \text{if } s \leqslant -2\epsilon\delta^2, \end{cases} \tag{4.3}$$

and define

$$\tilde{E}_j(t,x) = \begin{cases} \chi(\psi(t,x))E_j(t,x) & \text{if } (t,x) \in B_{2\delta}(0), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } j = 1,2,3. \tag{4.4}$$

Observe, that because of the definition of ψ in (4.2) and the definition of $\tilde{E}_j$ we have

$$\begin{aligned} \operatorname{supp} \tilde{E}_j &\subset \operatorname{supp} E_j \cap \operatorname{supp} \chi(\psi) \cap B_{2\delta}(0) \subset \{\varphi \leqslant 0\} \cap \{\psi \geqslant -2\epsilon\delta^2\} \cap B_{2\delta}(0) \\ &\subset \{\varphi \leqslant 0\} \cap \bigl\{\varphi - 2\epsilon\bigl(t^2 + |x|^2\bigr) > -2\epsilon\delta^2\bigr\} \subset B_\delta(0). \end{aligned} \tag{4.5}$$

Regarding the regularity of $\tilde{E}_j$ we have $\tilde{E}_j \in \mathcal{E}'(\mathbf{R}; H^1(\mathbf{R}^3))$ which implies $\tilde{E}_j \in H^{-N,1}$ for some N.

Using the fact that E solves Eq. (3.8) we will derive an equation for $\tilde{E}$. At first we notice that

$$P(x,D)\tilde{E}_j = P(x,D)\bigl(\chi(\psi)E_j\bigr) = \chi(\psi)P(x,D)E_j + \bigl[P(x,D), \chi(\psi)\bigr]E_j, \quad j = 1,2,3,$$

where $[P(x,D), \chi(\psi)]$ denotes the commutator of the operators $P(x,D)$ and $\chi(\psi)$. According to commutator rules this commutator is a first order operator. The commutator $[P(x,D), \chi(\psi)]E_j$ vanishes on $\chi(\psi) \equiv 1$ and $\chi(\psi) \equiv 0$. Taking (4.3) and (4.4) into consideration we obtain

$$\operatorname{supp}\bigl[P(x,D), \chi(\psi)\bigr]E_j \subset \{\psi \leqslant -\epsilon\delta^2\} \cap B_{2\delta}(0). \tag{4.6}$$

Furthermore,

$$\bigl[P(x,D), \chi(\psi)\bigr]E_j \in H^{-N,0}, \quad j = 1,2,3, \tag{4.7}$$

for some positive integer N.

Using (3.8) we obtain a system of equations for $\tilde{E}$. For brevity

$$Q_j(x,D)E(t,x) = \sum_{k,m=1}^{3} p_m^{jk}(x)\frac{\partial}{\partial x_k}E_m(t,x) + q_j^k(x)E_k(t,x), \quad j = 1,2,3,$$

and we have

$$\begin{aligned} P(x,D)\tilde{E}_j &= \chi(\psi)P(x,D)E_j + \bigl[P(x,D), \chi(\psi)\bigr]E_j \\ &= \chi(\psi)Q_j(x,D)E_j + \bigl[P(x,D), \chi(\psi)\bigr]E_j \\ &= Q_j(x,D)\tilde{E} - \bigl[Q_j(x,D), \chi(\psi)\bigr]E + \bigl[P(x,D), \chi(\psi)\bigr]E_j, \\ j &= 1,2,3. \end{aligned} \tag{4.8}$$

Similarly to (4.6) and (4.7) we have

$$\operatorname{supp}\big[Q_j(x,D),\chi(\psi)\big]E \subset \{\psi \leqslant -\epsilon\delta^2\} \cap B_{2\delta}(0) \quad \text{and}$$
$$\big[Q_j(x,D),\chi(\psi)\big]E \in H^{-N,1}. \tag{4.9}$$

Step 4: The Carleman estimate for the system. In the following c will denote a constant depending only on the surface function ψ and the operator $P(x,D)$. Further dependences will be stated explicitly. The size of the constant will be different at each particular instant.

Now we can apply the Carleman estimate (4.1) to each of the three equations in (4.8). Adding up the three estimates and the triangle inequality yields

$$\begin{aligned}
&\frac{1}{\tau}\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\tilde{E}\big|_{2,\tau}^2 \\
&\quad\leqslant c\Bigg(\sum_{j=1}^{3}\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}Q_j(x,D)\tilde{E}\big|_0^2 \\
&\qquad+\sum_{j=1}^{3}\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\big[Q_j(x,D),\chi(\psi)\big]E\big|_0^2+\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\big[P(x,D),\chi(\psi)\big]E\big|_0^2 \\
&\qquad+e^{-\delta\tau}|\tilde{E}|_{-N,1,\tau}^2+e^{-\delta\tau}\sum_{j=1}^{3}\big|\chi(\psi)Q_j(x,D)E\big|_{-N,0,\tau}^2 \\
&\qquad+e^{-\delta\tau}\sum_{j=1}^{3}\big|\big[Q_j(x,D),\chi(\psi)\big]E\big|_{-N,0,\tau}^2+e^{-\delta\tau}\big|\big[P(x,D),\chi(\psi)\big]E\big|_{-N,0,\tau}^2\Bigg).
\end{aligned} \tag{4.10}$$

Our goal is to simplify this inequality by estimating the terms on the right-hand side. We know that $Q_j(x,t)$ is a first order operator with respect to the space variables with smooth coefficient which implies

$$\sum_{j=1}^{3}\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}Q_j(x,D)\tilde{E}\big|_0^2 \leqslant c\sum_{j=1}^{3}\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\partial_j\tilde{E}\big|_0^2. \tag{4.11}$$

In order to treat this term further we will use the formula ([7] or [1, Lemma 5])

$$e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\partial_k u = \big(\partial_k - \tau\partial_k\psi - \partial_{kt}^2\psi\partial_t\big)e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}u, \quad k=0,1,2,3, \tag{4.12}$$

which can also be verified using the fact that ψ is a second degree polynomial. Hence, together with (4.11),

$$\sum_{j=1}^{3}\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}Q_j(x,D)\tilde{E}_j\big|_0^2 \leqslant c\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\tilde{E}\big|_{1,\tau}^2. \tag{4.13}$$

Next we analyze the second and the third term on the right-hand side in (4.10). Using Parseval's identity twice we obtain

$$\sum_{j=1}^{3}\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\big[Q_j(x,D),\chi(\psi)\big]E\big|_0^2+\big|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\big[P(x,D),\chi(\psi)\big]E\big|_0^2$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^4}\sum_{j=1}^{3}\bigl|e^{-\frac{1}{2\tau}\xi_0^2}\mathcal{F}\bigl[e^{\tau\psi}\bigl[Q_j(x,D),\chi(\psi)\bigr]E\bigr]\bigr|_0^2 \\
&\quad + \frac{1}{(2\pi)^4}\bigl|e^{-\frac{1}{2\tau}\xi_0^2}\mathcal{F}\bigl[e^{\tau\psi}\bigl[P(x,D),\chi(\psi)\bigr]E\bigr]\bigr|_0^2 \\
&= \frac{1}{(2\pi)^4}\sum_{j=1}^{3}\bigl|e^{-\frac{1}{2\tau}\xi_0^2}\bigl(\tau^2+\xi_0^2\bigr)^N\mathcal{F}\bigl[e^{\tau\psi}\bigl[Q_j(x,D),\chi(\psi)\bigr]E\bigr]\bigr|_{-N,0,\tau}^2 \\
&\quad + \frac{1}{(2\pi)^4}\bigl|e^{-\frac{1}{2\tau}\xi_0^2}\bigl(\tau^2+\xi_0^2\bigr)^N\mathcal{F}\bigl[e^{\tau\psi}\bigl[P(x,D),\chi(\psi)\bigr]E\bigr]\bigr|_{-N,0,\tau}^2 \\
&\leqslant c\tau^{2N}\left(\sum_{j=1}^{3}\bigl|e^{\tau\psi}\bigl[Q_j(x,D),\chi(\psi)\bigr]E\bigr|_{-N,0,\tau}^2 + \bigl|e^{\tau\psi}\bigl[P(x,D),\chi(\psi)\bigr]E\bigr|_{-N,0,\tau}^2\right),
\end{aligned}
\tag{4.14}
$$

where we also used the fact that

$$\sup_{\xi\in\mathbf{R}^4} e^{-\frac{1}{2\tau}\xi_0^2}\bigl(\tau^2+\xi_0^2\bigr)^N = \tau^{2N}.$$

Now we apply the inequality [1, Lemma 1]

$$|fu|_{-N,0,\tau} \leqslant c|u|_{-N,0,\tau}\sum_{|\alpha|\leqslant N}\tau^{-|\alpha|}\|D^\alpha f\|_{C(\operatorname{supp} u)} \quad \text{for } u\in H^{-N,0},\ f\in C^N,$$

to the last two terms in (4.14) and obtain

$$
\begin{aligned}
&\sum_{j=1}^{3}\bigl|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\bigl[Q_j(x,D),\chi(\psi)\bigr]E\bigr|_0^2 + \bigl|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\bigl[P(x,D),\chi(\psi)\bigr]E\bigr|_0^2 \\
&\quad\leqslant c\tau^{2N}\sum_{|\alpha|\leqslant N}\tau^{-2|\alpha|}\|D^\alpha e^{\tau\psi}\|_{C(\{\psi\leqslant-\epsilon\delta^2\}\cap B_{2\delta}(0))}^2\bigl|\bigl[Q_j(x,D),\chi(\psi)\bigr]E\bigr|_{-N,0,\tau}^2 \\
&\qquad + c\tau^{2N}\sum_{|\alpha|\leqslant N}\tau^{-2|\alpha|}\|D^\alpha e^{\tau\psi}\|_{C(\{\psi\leqslant-\epsilon\delta^2\}\cap B_{2\delta}(0))}^2\bigl|\bigl[P(x,D),\chi(\psi)\bigr]E\bigr|_{-N,0,\tau}^2 \\
&\quad\leqslant c\tau^{2N}e^{-\epsilon\delta^2\tau},
\end{aligned}
\tag{4.15}
$$

where we used (4.6), (4.7), and (4.9) as well. Of course the constant c depends on E.

Finally, the last four terms in (4.10) are bounded by $ce^{-\delta\tau}$ with the constant c again depending on E. Hence, taking (4.13) and (4.15) into consideration, (4.10) becomes

$$\frac{1}{\tau}\bigl|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\tilde{E}\bigr|_{2,\tau}^2 \leqslant c_1\bigl|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\tilde{E}\bigr|_{1,\tau}^2 + c_2\bigl(\tau^{2N}e^{-\epsilon\delta^2\tau}+e^{-\delta\tau}\bigr),$$

where the constant c_2 depends also on E. Choosing $\tau > 2c_1$ and observing that $|u|_{1,\tau}^2 \leqslant 1/\tau^2|u|_{2,\tau}^2$ the first term on the right-hand side can be absorbed into the left-hand side

$$\frac{1}{2\tau}\bigl|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\tilde{E}\bigr|_{2,\tau}^2 \leqslant c_2\bigl(\tau^{2N}e^{-\epsilon\delta^2\tau}+e^{-\delta\tau}\bigr).$$

Furthermore, the two exponential expression on the right-hand side can be combined

$$\frac{1}{\tau}\left|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\tilde{E}\right|_{2,\tau}^2 \leqslant 2c_2e^{-\beta\tau},$$

where $\beta = \min\{\epsilon\delta^2, \delta\}/2$. Using the properties of the norm on the left-hand side yields

$$\left|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}\tilde{E}\right|_0^2 \leqslant \frac{2c_2}{\tau^3}e^{-\beta\tau}. \tag{4.16}$$

We conclude by applying Proposition 4.1 in [7] which states that

$$\left|e^{-\frac{1}{2\tau}D_t^2}e^{\tau\psi}u\right|_0 \to 0 \quad \text{as } \tau \to \infty$$

implies $u \equiv 0$ on the set $\{\psi > 0\}$. Hence $\tilde{E} \equiv 0$ in $\{\psi > -\beta/2\}$. This implies that $E \equiv 0$ in a neighborhood of $0 = (t_0, x_0)$. $\square$

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